Part 3.3 Differentiation v1 2019-20

Taylor Polynomials

Definition 3.3.1 *(Taylor 1715 and Maclaurin 1742) If a is a fixed number, and f is a function whose first n derivatives exist at a then the* ***Taylor polynomial of degree** n **for** f **at** a is*

$$T_{n,a}f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ... + \frac{f^{(n)}(a)}{n!}(x-a)^n .$$

Alternatively,

$$T_{n,a}f(x) = \sum_{r=0}^{n} \frac{f^{(r)}(a)}{r!}(x-a)^r ,$$

where $f^{(0)}(x) = f(x)$.

Though it may appear daunting to calculate all these derivatives, if the function f has a trigonometric factor there is often a pattern in the derivatives that can be exploited.

Example 3.3.2 *Calculate*

$$T_{8,0}\left(e^x \sin x\right).$$

Solution If $f(x) = e^x \sin x$ then

$$\begin{aligned} f(x) &= e^x \sin x \\ f^{(1)}(x) &= e^x \sin x + e^x \cos x \\ f^{(2)}(x) &= e^x \sin x + e^x \cos x + e^x \cos x - e^x \sin x \\ &= 2e^x \cos x \\ f^{(3)}(x) &= 2e^x \cos x - 2e^x \sin x \\ f^{(4)}(x) &= 2e^x \cos x - 2e^x \sin x - 2e^x \sin x - 2e^x \cos x \\ &= -4e^x \sin x. \end{aligned}$$

The important observation is that $f^{(4)}(x) = -4f(x)$, for this means that

$$f^{(5)}(x) = -4f^{(1)}(x)\,,\ f^{(6)}(x) = -4f^{(2)}(x)\,,\ f^{(7)}(x) = -4f^{(3)}(x)$$

and

$$f^{(8)}(x) = -4f^{(4)}(x) = 16f(x)\,.$$

Thus

$$\begin{aligned} f(0) &= 0, f^{(1)}(0) = 1, f^{(2)}(0) = 2, f^{(3)}(0) = 2, f^{(4)}(0) = 0, \\ f^{(5)}(0) &= -4, f^{(6)}(0) = -8, f^{(7)}(0) = -8, f^{(8)}(0) = 0. \end{aligned}$$

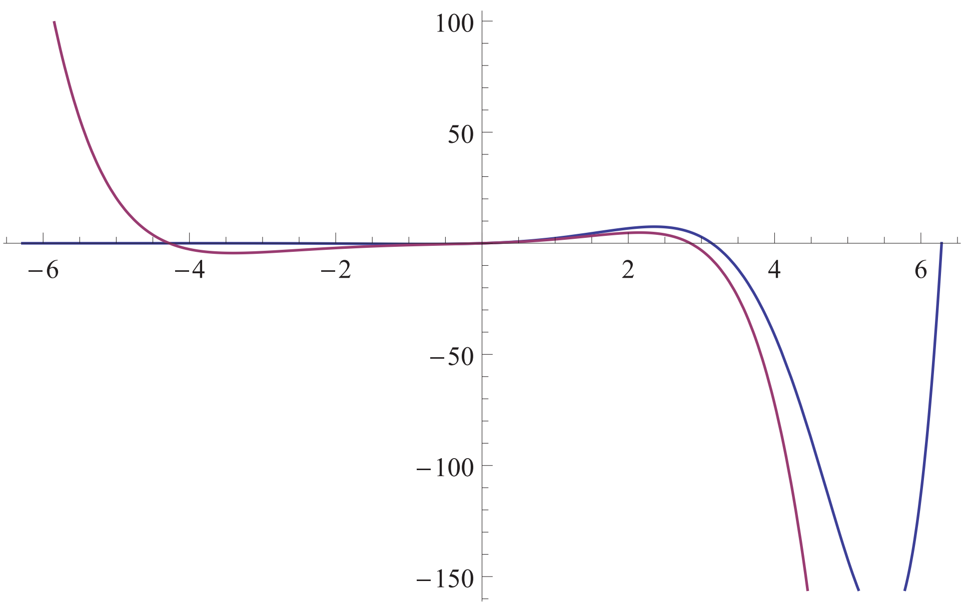
Hence

$$\begin{aligned} T_{8,0}\left(e^x \sin x\right) &= 0 + x + 2\frac{x^2}{2!} + 2\frac{x^3}{3!} + 0\frac{x^4}{4!} - 4\frac{x^5}{5!} - 8\frac{x^6}{6!} - 8\frac{x^7}{7!} + 0\frac{x^8}{8!} \\ &= x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} - \frac{x^6}{90} - \frac{x^7}{630}. \end{aligned}$$

■

Note When the function f contains trigonometric functions we often find a relationship between f and $f^{(4)}$ as we saw above. Such relations should always be exploited to reduce work.

Illustrating Example 3.3.2 the blue line is $e^x \sin x$, the red line is $T_{8,0}\left(e^x \sin x\right)$.



This pattern in derivatives can be seen again in

Example 3.3.3

$$T_{4,0}\left(\cos^2 x\right) = 1 - x^2 + \frac{1}{3}x^4.$$

Solution Let $f(x) = \cos^2 x$. Then $f^{(1)}(x) = -2\cos x \sin x = -\sin 2x$. It is important to write it in this way because, continuing,

$$f^{(2)}(x) = -2\cos 2x \quad \text{and} \quad f^{(3)}(x) = 4\sin 2x = -4f^{(1)}(x)\,.$$

This relation between third and first derivatives means that $f^{(n)}(x) = -4f^{(n-2)}(x)$ for all $n \geq 3$ which simplifies the calculations of

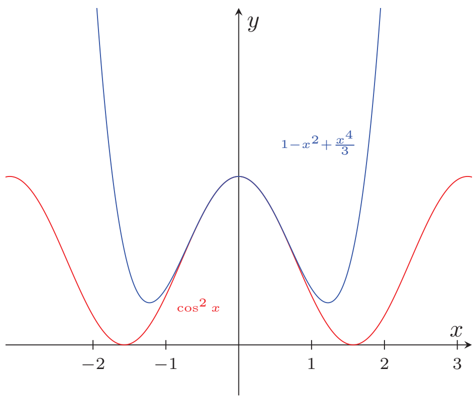
$$f(0) = 1,\ f^{(1)}(0) = 0, f^{(2)}(0) = -2, f^{(3)}(0) = -4f^{(1)}(0) = 0,$$

and $f^{(4)}(0) = -4f^{(2)}(0) = 8$. Thus

$$T_{4,0}\left(\cos^2 x\right) = 1 + 0x - 2\frac{x^2}{2!} + 0\frac{x^3}{3!} + 8\frac{x^4}{4!} = 1 - x^2 + \frac{x^4}{3}.$$

■

Illustrating Example 3.3.3.



The next example illustrates a method which can often be applied when f is a quotient.

Example 3.3.4 *With*

$$f(x) = \frac{e^x}{1+x}$$

calculate $T_{5,0}f(x)$.

Solution Because differentiating quotients leads to complicated expressions we yet again follow the principle of ridding ourselves of fractions by multiplying up as

$$(1+x)\,f(x) = e^x.$$

Then repeated differentiation gives

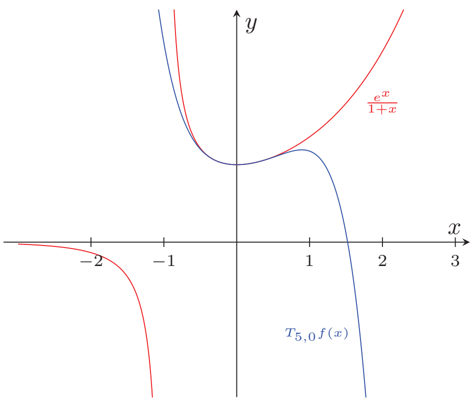
$$\begin{aligned}
(1+x)\, f'(x) + f(x) &= e^x, \quad \text{thus } f'(0)+f(0)=1. \\
(1+x)\, f''(x) + 2f'(x) &= e^x, \quad \text{thus } f''(0)+2f'(0)=1. \\
(1+x)\, f^{(3)}(x) + 3f''(x) &= e^x, \quad \text{thus } f^{(3)}(0)+3f''(0)=1. \\
(1+x)\, f^{(4)}(x) + 4f^{(3)}(x) &= e^x, \quad \text{thus } f^{(4)}(0)+4f^{(3)}(0)=1. \\
(1+x)\, f^{(5)}(x) + 5f^{(4)}(x) &= e^x, \quad \text{thus } f^{(5)}(0)+5f^{(4)}(0)=1.
\end{aligned}$$

Starting from $f(0)=1$ we can solve to get $f'(0)=0$, $f''(0)=1$, $f^{(3)}(0)=-2$, $f^{(4)}(0)=9$ and $f^{(5)}(0)=-44$. Then

$$\begin{aligned}
T_{5,0}\left(\frac{e^x}{1+x}\right) &= 1+0x+1\frac{x^2}{2!}-2\frac{x^3}{3!}+9\frac{x^4}{4!}-44\frac{x^5}{5!} \\
&= 1+\frac{1}{2}x^2-\frac{1}{3}x^3+\frac{3}{8}x^4-\frac{11}{30}x^5.
\end{aligned}$$

■

Illustrating Example 3.3.4



Questions; how well does $T_{n,a}f(x)$ approximate $f(x)$, does $T_{n,a}f(x)$ converge as $n \to \infty$ and, if it does, does it converge to $f(x)$? These questions can be answered by studying the difference $f(x) - T_{n,a}f(x)$.

Definition 3.3.5 *The* ***Remainder****,* $R_{n,a}f(x)$*, is defined by*

$$R_{n,a}f(x) = f(x) - T_{n,a}f(x)\,. \tag{1}$$

Note that when $t = x$ in the definition of $T_{n,t}f(x)$ we get

$$\begin{aligned} T_{n,x}f(x) &= f(x) + f'(x)(x-x) + \frac{f''(x)}{2!}(x-x)^2 + ... + \frac{f^{(n)}(x)}{n!}(x-x)^n \\ &= f(x). \end{aligned} \tag{2}$$

Thus the remainder can be written as

$$R_{n,a}f(x) = T_{n,x}f(x) - T_{n,a}f(x).$$

So we are looking at the difference of a function of t, namely $T_{n,t}f(x)$, at $t = x$ and $t = a$. With an application of the Mean Value Theorem in mind, this makes us ask how $T_{n,t}f(x)$ changes as t varies.

We start with quite an amazing result, that the derivative w.r.t t of the polynomial $T_{n,t}f(x)$ should be so simple!

Lemma 3.3.6 *If the first $n+1$ derivatives of f exist on an open neighbourhood of x then*

$$\frac{d}{dt}T_{n,t}f(x) = \frac{(x-t)^n}{n!}f^{(n+1)}(t),$$

for all t in the open neighbourhood.

Proof in the lectures observes at one point that a term from one bracket in a series cancels a term in the next bracket. Here we give a more formal proof, based on manipulating series.

By definition

$$T_{n,t}f(x) = \sum_{r=0}^{n} \frac{f^{(r)}(t)}{r!}(x-t)^r.$$

This is differentiable w.r.t t if, and only if, every $f^{(r)}$, $0 \le r \le n$ is differentiable. Yet $f^{(i+1)}$ differentiable implies $f^{(i)}$ differentiable so $T_{n,t}f$ is differentiable if, and only if, $f^{(n)}$, is differentiable, that is, f is n+1 times differentiable. Since we are assuming this we can continue:

$$\begin{aligned}
\frac{d}{dt}T_{n,t}f(x) &= \frac{d}{dt}\sum_{r=0}^{n}\frac{f^{(r)}(t)}{r!}(x-t)^r \\
&= \frac{d}{dt}\left(f(t)+\sum_{r=1}^{n}\frac{f^{(r)}(t)}{r!}(x-t)^r\right) \\
&= f^{(1)}(t)+\sum_{r=1}^{n}\left(\frac{f^{(r+1)}(t)}{r!}(x-t)^r-\frac{f^{(r)}(t)}{(r-1)!}(x-t)^{r-1}\right) \\
&= f^{(1)}(t)+\sum_{r=1}^{n}\frac{f^{(r+1)}(t)}{r!}(x-t)^r-\sum_{r=1}^{n}\frac{f^{(r)}(t)}{(r-1)!}(x-t)^{r-1}.
\end{aligned}$$

In the second sum we change variable from r to $r-1$, which we then relabel as r, so r now runs from 0 to $n-1$. Thus

$$\begin{aligned}
\frac{d}{dt}T_{n,t}f(x) &= f^{(1)}(t)+\sum_{r=1}^{n}\frac{f^{(r+1)}(t)}{r!}(x-t)^r-\sum_{r=0}^{n-1}\frac{f^{(r+1)}(t)}{r!}(x-t)^r \\
&= f^{(1)}(t)+\left(\sum_{r=1}^{n-1}\frac{f^{(r+1)}(t)}{r!}(x-t)^r+\frac{f^{(n+1)}(t)}{n!}(x-t)^n\right) \\
&\qquad -\left(\sum_{r=1}^{n-1}\frac{f^{(r+1)}(t)}{r!}(x-t)^r+f^{(1)}(t)\right) \\
&= \frac{f^{(n+1)}(t)}{n!}(x-t)^n.
\end{aligned}$$

■

An application of the Mean Value Theorem gives

Theorem 3.3.7 *Taylor's Theorem with* ***Cauchy's form*** *of the error. If the first $n+1$ derivatives of f exist on an open interval containing a and x then*

$$R_{n,a}f(x)=\frac{f^{(n+1)}(c)}{n!}(x-c)^n(x-a) \tag{3}$$

for some c between a and x.

Proof Consider

$$\begin{aligned}\frac{R_{n,a}f(x)}{x-a} &= \frac{f(x)-T_{n,a}f(x)}{x-a} \quad \text{by definition of } R_{n,a}, \\ &= \frac{T_{n,x}f(x)-T_{n,a}f(x)}{x-a}.\end{aligned}$$

by (2). Let $h(t) = T_{n,t}f(x)$ so we can rewrite the last equality as

$$\frac{R_{n,a}f(x)}{x-a} = \frac{h(x)-h(a)}{x-a} = h'(c)\,,$$

for some c between a and x by the Mean Value Theorem applied to h. Continuing

$$h'(c) = \frac{d}{dt}T_{n,t}f(x)\Big|_{t=c} = \frac{(x-c)^n}{n!}f^{(n+1)}(c)\,,$$

by Lemma. ■

This result has a weakness in that the unknown c occurs in **two** terms on the right hand side. Strange that Cauchy's error was derived using the Mean Value Theorem; what would follow from Cauchy's Mean Value Theorem? Recall *Cauchy*'s Mean Value Theorem, that if g, h are continuous on $[a,b]$, differentiable on (a,b) and $g'(x) \neq 0$ for all $x \in (a,b)$ then there exists $c \in (a,b)$ such that

$$\frac{h(b)-h(a)}{g(b)-g(a)} = \frac{h'(c)}{g'(c)}.$$

An argument based on this gives

Theorem 3.3.8 *Taylor's Theorem with* ***Lagrange's form*** *of the error (1797). If the first $n+1$ derivatives of f exist on an open interval containing a and x then*

$$R_{n,a}f(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \tag{4}$$

for some c between a and x.

Proof Consider x to be fixed. As in previous proof let $h(t) = T_{n,t}f(x)$ and g to be chosen but continuous on $[a,x]$, differentiable on (a,x) and with $g'(t) \neq 0$ for all $t \in (a,x)$. Then

$$\begin{aligned}
\frac{R_{n,a}f(x)}{g(x)-g(a)} &= \frac{T_{n,x}f(x)-T_{n,a}f(x)}{g(x)-g(a)} \quad \text{as in above proof,}\\
&= \frac{1}{g'(c)}\frac{d}{dt}T_{n,t}f(x)\Big|_{t=c} \quad \text{by Cauchy's M. V. Theorem,}\\
&= \frac{(x-c)^n}{n!g'(c)}f^{(n+1)}(c)\,,
\end{aligned}$$

by Lemma. If we choose $g'(t) = (x-t)^n$ then

$$\frac{R_{n,a}f(x)}{g(x)-g(a)} = \frac{(x-c)^n}{n!\,(x-c)^n}f^{(n+1)}(c) = \frac{f^{(n+1)}(c)}{n!},$$

which multiplies up to give

$$R_{n,a}f(x) = (g(x)-g(a))\,\frac{f^{(n+1)}(c)}{n!}.$$

The right hand side now only contains **one** occurrence of the unknown c, as required. Integrate this choice of g' to get

$$g(x)-g(a) = \int_a^x g'(t)\,dt = \frac{(x-a)^{n+1}}{n+1}.$$

Thus

$$R_{n,a}f(x) = (g(x)-g(a))\,\frac{f^{(n+1)}(c)}{n!} = \frac{(x-a)^{n+1}}{(n+1)}\frac{f^{(n+1)}(c)}{n!}$$

as required. ■

In Theorem 3.3.8 we now have only one occurrence of the unknown c, along with a larger denominator. If we set $h = x-a$ in Taylor's Theorem with Lagrange's error we get

$$\begin{aligned}
f(a+h) &= f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + ... + \frac{h^n}{n!}f^{(n)}(a) +\\
&\quad + \frac{h^{n+1}}{(n+1)!}f^{(n+1)}(a+\theta h)
\end{aligned}$$

for some $0 < \theta < 1$.

Taylor's Theorem is often used in *Maclaurin's Form* which simply has $a = 0$:

$$f(x) = \sum_{r=0}^{n} \frac{f^{(r)}(0)}{r!} x^r + \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1},$$

for some c between 0 and x.

As a first application of how well $T_{n,a}f(x)$ approximates $f(x)$,

Example 3.3.9

$$\left| e^x \sin x - \left(x + 2\frac{x^2}{2!} + 2\frac{x^3}{3!} \right) \right| \leq \frac{e^{|c|}}{6} x^4.$$

for some c between 0 and x.

Solution in Tutorial Note that from the workings of Example 3.3.2,

$$T_{3,0}\left(e^x \sin x\right) = 0 + x + 2\frac{x^2}{2!} + 2\frac{x^3}{3!}, \quad \text{and} \quad f^{(4)}(x) = -4e^x \sin x.$$

Then

$$\left| e^x \sin x - \left(x + 2\frac{x^2}{2!} + 2\frac{x^3}{3!} \right) \right| = \frac{4\left|e^c \sin c\right|}{4!} x^4 \leq \frac{e^{|c|}}{6} x^4.$$

for some c between 0 and x. ■

You can, in fact, improve this result because $f^{(4)}(0) = 0$. For then $T_{3,0}\left(e^x \sin x\right) = T_{4,0}\left(e^x \sin x\right)$. Thus

$$\left| e^x \sin x - \left(x + 2\frac{x^2}{2!} + 2\frac{x^3}{3!} \right) \right| = \frac{\left|f^{(5)}(c)\right|}{5!} |x|^5 .$$

Now note that $f^{(4)}(x) = -4e^x \sin x$ implies $f^{(5)}(x) = -4\left(e^x \sin x + e^x \cos x\right)$. So $\left|f^{(5)}(c)\right| \leq 8e^c$, and thus

$$\left| e^x \sin x - \left(x + 2\frac{x^2}{2!} + 2\frac{x^3}{3!} \right) \right| \leq \frac{e^{|c|}}{15} |x|^5 .$$

Aside, one can do better than $|\sin x + \cos x| \leq |\sin x| + |\cos x| \leq 2$ by noting that $\sin x + \cos x = y + \sqrt{1 - y^2}$ where $y = \sin x$. Look for turning points (at $y = \pm\sqrt{4/5}$) and thus a maximum of $\sqrt{4/5} + \sqrt{1/5} \approx 1.34...$

Example 3.3.10 *Use Lagrange's form for the error to show that*

$$\left|\cos^2 x - \left(1 - x^2 + \frac{1}{3}x^4\right)\right| \leq \frac{2}{15}\left|x\right|^5.$$

Hence show that

$$\lim_{x\to 0} \frac{\cos^2 x - 1 + x^2}{x^4} = \frac{1}{3}.$$

Solution From Example 3.3.3 we have

$$T_{4,0}\left(\cos^2 x\right) = 1 - x^2 + \frac{1}{3}x^4.$$

Thus

$$\begin{aligned}\left|\cos^2 x - \left(1 - x^2 + \frac{1}{3}x^4\right)\right| &= \left|\cos^2 x - T_{4,0}\left(\cos^2 x\right)\right| \\ &= \left|R_{4,0}\left(\cos^2 x\right)\right| \\ &= \left|\frac{f^{(5)}(c)}{5!}x^5\right|,\end{aligned}$$

for some c between 0 and x, by Lagrange's error. As seen previously,

$$f^{(5)}(x) = -4f^{(3)}(x) = 16f^{(1)}(x) = -16\sin 2x.$$

Thus

$$\left|\frac{f^{(5)}(c)}{5!}x^5\right| = \frac{16}{5!}\left|\sin 2c\right|\left|x\right|^5 \leq \frac{2}{15}\left|x\right|^5.$$

This gives the first stated result. For the second, divide through by x^4 to get

$$\left|\frac{\cos^2 x - 1 + x^2}{x^4} - \frac{1}{3}\right| \leq \frac{2}{15}\left|x\right|.$$

This can be opened out as

$$\frac{1}{3} - \frac{2}{15}\left|x\right| \leq \frac{\cos^2 x - 1 + x^2}{x^4} \leq \frac{1}{3} + \frac{2}{15}\left|x\right|$$

Let $x \to 0$ and quote the Sandwich Rule to get result. ■

Aside, we see that $f^{(5)}(0) = 0$ which means that $T_{4,0}(\cos^2 x) = T_{5,0}(\cos^2 x)$. Then

$$\left|\cos^2 x - \left(1 - x^2 + \frac{1}{3}x^4\right)\right| = \left|R_{5,0}\left(\cos^2 x\right)\right| = \left|\frac{f^{(6)}(c)}{6!}x^6\right|.$$

Now $f^{(6)}(x) = 16 f^{(2)}(x) = -32\cos 2x$. Thus

$$\left|\cos^2 x - \left(1 - x^2 + \frac{1}{3}x^4\right)\right| = \frac{32}{6!}\left|\cos 2c\right| \left|x\right|^6 \leq \frac{2}{45}\left|x\right|^6.$$

This is an improvement over the previous bound as long as $|x| < 3$.

Taylor Series

Definition 3.3.11 *If all the higher derivatives of f exist in some neighbourhood of $a \in \mathbb{R}$ then the* ***Taylor Series of*** f ***at*** a *is*

$$\sum_{r=0}^{\infty} \frac{f^{(r)}(a)}{r!} (x-a)^r . \tag{5}$$

There are two immediate questions.

Question 1. Does the series converge?

This series trivially converges at $x = a$. Being a power series we can use tests from MATH10242, such as the Ratio Test or Comparison Tests to find an $R \geq 0$ for which

- if $|x-a| < R$ then the series converges at this x,
- if $|x-a| > R$ then the series diverges at this x,
- while if $|x-a| = R$ the series may, or may not converge at such x.

Here R is called the *radius of convergence*. It is possible that $R = \infty$, for example the series for e^x. It is possible that $R = 0$; an example was given by Lerch in 1888 of a function well-defined on $\mathbb{R}$ yet whose Taylor series diverges for all $x \neq 0$.

Question 2. If the Taylor Series of f converges at $x \in \mathbb{R}$ does it converge to $f(x)$?

Even if $R > 0$ and $x_0 \neq a$ is in the interval of convergence, there is no assurance that the value of the series at x_0 equals $f(x_0)$. This is the case with Cauchy's example (1823), of

$$f(x) = e^{-1/x^2} \text{ for } x \neq 0 \text{ with } f(0) = 0.$$

I leave this as a (hard) exercise for students. You will have to calculate $f^{(n)}(0)$ for each $n \geq 1$ by first principles, using the limit definition. It can be shown in this way that $f^{(n)}(0) = 0$ for all $n \geq 1$. Thus the Taylor Series for $f(x)$ is

$$0 + 0x + 0\frac{x^2}{2!} + 0\frac{x^3}{3!} + ...$$

which converges for all $x \in \mathbb{R}$. But it's sum is $f(x)$ **only** when $x = 0$.

Yet Question 2 does have an answer: Recall from the first year course, Sequences and Series, an infinite series is defined to be the limit of the sequence of partial sums, if the limit exists. Thus

$$\sum_{r=0}^{\infty} \frac{f^{(r)}(a)}{r!} (x-a)^r = \lim_{n\to\infty} \sum_{r=0}^{n} \frac{f^{(r)}(a)}{r!} (x-a)^r = \lim_{n\to\infty} T_{n,a}f(x)\,.$$

So the Taylor Series of f converges to f for those x for which

$$\lim_{n\to\infty} T_{n,a}f(x) = f(x)\,.$$

This can be rearranged to $\lim_{n\to\infty} (f(x) - T_{n,a}f(x)) = 0$, the same as

$$\lim_{n\to\infty} R_{n,a}f(x) = 0.$$

In most cases the limit of the remainder term as $n \to \infty$ will make use of the following result.

Lemma 3.3.12 *For any $y \in \mathbb{R}$ we have*

$$\lim_{n\to 0} \frac{y^n}{n!} = 0. \tag{6}$$

Proof It is a result from First Year Sequences and Series that $\{y^n/n!\}_{n\geq 1}$ is a null sequence.

It can be noted though that we have been assuming this result implicitly in this course. We have defined e^y by the infinite series $\sum_{r=0}^{n} y^r/r!$. Yet if an infinite series converges its terms must tend to zero, which is the statement of this Lemma. ■

Application Consider Lagrange's form of the error term

$$R_{n,0}f(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}.$$

Assume we have a bound on the derivatives of f of the form

$$\left|f^{(n)}(x)\right| \leq g\,(x)\, C^n \tag{7}$$

for some constant $C > 0$, and positive function $g(x)$, for all $n \geq 1$. Then

$$|R_{n,0}f(x)| \leq g(c)\, \frac{(C\,|x|)^{n+1}}{(n+1)!} \to 0$$

as $n \to \infty$, by (6), for the x for which (7) holds. That is, for such x, $R_{n,0}f(x) \to 0$, i.e. $T_{n,0}f(x) \to f(x)$ as $n \to \infty$.

Example 3.3.13 *Find the Taylor Series for* $\cos^2 x$ *around* $a = 0$ *and show that the series converges to* $\cos^2 x$ *for all* $x \in \mathbb{R}$*.*

Solution If $f(x) = \cos^2 x$ then $f^{(1)}(x) = -2\cos x \sin x = -\sin(2x)$ and, as seen before, $f^{(n)}(x) = -4f^{(n-2)}(x)$ for all $n \geq 3$.

If n is odd then $f^{(n)}(0)$ will be a multiple of $f^{(1)}(0) = 0$. So the only non-zero terms will come from even n.

If $n = 2r$ then

$$f^{(n)}(x) = (-4)^{r-1} f^{(2)}(x) = -2(-4)^{r-1} \cos 2x = (-1)^r 2^{n-1} \cos 2x.$$

Thus $f^{(n)}(0) = (-1)^r 2^{n-1}$ if $n = 2r$ is even.

Hence the Taylor Series for $\cos^2 x$ is

$$1 + \sum_{\substack{n=1 \\ n=2r \text{ even}}}^{\infty} (-1)^r 2^{n-1} \frac{x^n}{n!} = 1 + \sum_{r=1}^{\infty} (-1)^r 2^{2r-1} \frac{x^{2r}}{(2r)!}.$$

For what x does $\lim_{n\to\infty} R_{n,0}(\cos^2 x) = 0$? By Lagrange's form of the error

$$R_{n,0}\left(\cos^2 x\right) = f^{(n+1)}(c)\,\frac{x^{n+1}}{(n+1)!}$$

for some c between 0 and x. In the present case, we look at the modulus so we don't worry about the sign, when

$$\left|f^{(n+1)}(c)\right| = \begin{cases} 2^n \left|\cos 2c\right| & \text{if } n+1 \text{ is even,} \\ 2^{n+1}\left|\sin 2c\right| & \text{if } n+1 \text{ is odd,} \end{cases}$$

Then $\left|f^{(n+1)}(c)\right| \leq 2^{n+1}$ in both cases. Thus,

$$\left|R_{n,0}\left(\cos^2 x\right)\right| \leq \frac{(2|x|)^{n+1}}{(n+1)!} \to 0$$

as $n \to \infty$ for **any** $x \in \mathbb{R}$ by the Lemma above,. Hence the Taylor Series for $\cos^2 x$ converges to $\cos^2 x$ for all $x \in \mathbb{R}$. ■

Example 3.3.14 *The Taylor Series for* $(1+x)^t$ *for* $t \in \mathbb{R}$*, is*

$$\sum_{r=0}^{\infty} \frac{t(t-1)\dots(t-r+1)}{r!} x^r,$$

and this converges to $(1+x)^t$ *when* $-1 < x < 1$*. This is a generalisation of the Binomial Theorem.*

Solution see Appendix.

Example 3.3.15 *The Taylor Series for* $\ln(1+x)$ *is*

$$\sum_{r=1}^{\infty} \frac{(-1)^{r-1}\, x^r}{r},$$

and this converges to $\ln(1+x)$ *when* $-1 < x \leq 1$. *If we put* $x = 1$ *we get*

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots\ . \tag{8}$$

Solution see Appendix.